

specimens tested consisted of approximately square arrays of 78 cells with five partially unsupported cells on two sides of each specimen (Fig. 1). Thus, in these arrays, 12.8% of the cells were not constrained in the same manner as interior cells. If specimens with exterior dimensions twice the size of those tested had been used, only 6.8% of the cells in the array would have been partially constrained. No attempt was made to evaluate the magnitude of this scaling factor, because of the apparently small effect it had on the total crushing stress. A limited amount of published test data was obtained for a 5052 H-39 aluminum hexagonal cell structure, shown in Fig. 12 along with the derived relations using the 5052 H-39 material properties.

Of additional interest, it was found that there is an upper limit on the  $t/S$  ratio at which the hexagonal cell structure will collapse in a mode of failure that is entirely different from the mode normally observed. This failure mode is essentially a gross shear failure, as shown in the two views in Fig. 13. The  $t/S$  ratio for the test specimen at which this mode of failure begins to appear under static loading is approximately 0.040. The energy absorption of the shear mode is less than would be obtained by the crushing mode previously considered for corresponding  $t/S$  ratios. The emergency of the shear mode is controlled by the material properties and the  $t/S$  ratio.

### VII. Conclusion

The foregoing analysis is presented as a first approach to the problem of determining the mean crushing stress of

hexagonal cell structures. As previously stated, the purpose of this analysis was to derive a relationship for calculating the mean crushing stress and to determine the parameters that control it, with the end goal of providing a rational basis for the design of more efficient energy-absorbing structures. The method of analysis is a standard one, used to a large extent in structural analyses. There are a number of refinements to this theory, but no attempt was made to apply them because of the complex nature of the structure considered.

The results indicate a favorable correlation between theory and experiment for the limited amount of test data available. The use of higher yield stress materials, possessing adequate ductility, for the fabrication of the hexagonal cell structures would result in energy-absorbing properties well in excess of those exhibited by present-day aluminum-alloy cell structures.

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## Stretching of a Polar-Orthotropic Disk of Varying Thickness under Arbitrary Body Forces

CHARLES W. BERT\* AND FRANCIS W. NIEDENFUHR†  
*Ohio State University, Columbus, Ohio*

General equations are formulated for the elastic behavior of circular disks with radial variations of thickness and of polar-orthotropic elastic moduli and subject to any arbitrary systems of in-plane boundary or body forces. These equations are applied to and solved for an annular disk with a power-function variation in stiffness. The analysis is applicable to the design of turbine disks. Numerical results are obtained for the following examples of homogeneous disks rotating about eccentric normal axes: 1) an isotropic, uniform-thickness disk; and 2) a polar-orthotropic disk in which the thickness varies inversely with the radius.

### Introduction

AMONG the earliest solutions of a plane elasticity problem involving nonsymmetric body forces is the analysis by Michell in 1900.<sup>1</sup> The first analysis of a varying-thickness disk subject to nonsymmetric generalized plane stress

but no body forces was probably that of Shepherd,<sup>2</sup> who in 1933 treated a straight-tapered, rectangular-planform disk. Recently, Musick<sup>3</sup> and Conway<sup>4</sup> independently considered circular disks of hyperbolic profile subject to nonsymmetric boundary forces.

One of the first solutions for a circular disk under gravity loading in its plane was that of Michell.<sup>1</sup> In 1935 Biot<sup>5</sup> noted that body forces derivable from plane potential functions do not appear in the plane equilibrium equations. Circular disks with nonsymmetric body forces *not* derivable from potential functions, but derivable from more general functions, probably were analyzed first in 1938 by Mindlin,<sup>6</sup> who treated an eccentrically rotating, uniform-thickness disk. Apparently, varying-thickness disks subject to nonsymmetric body forces were treated first in 1952 by Vainberg,<sup>7</sup> who analyzed a disk rotating about a diameter.

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\* Formerly Instructor, Department of Engineering Mechanics; now Program Director, Solid and Structural Mechanics Research, Battelle Memorial Institute, Columbus, Ohio. Member AIAA.

† Professor, Department of Engineering Mechanics.

**Table 1**  $C_1$  and  $D_1$  stress-function terms

Case	Remarks on a complementary solution of Eq. (8)
$n = 0$	The $C_1$ and $D_1$ terms are both independent solutions, i.e., $C_1$ and $D_1$ are both arbitrary.
$n = 1 - (e/\nu)$	The $C_1$ term is an independent solution and the $D_1$ term is not, i.e., $C_1$ is arbitrary and $D_1 = 0$ .
$n = \frac{1 + e + 2c - 2\nu}{1 - \nu}$	The $D_1$ term is an independent solution and the $C_1$ is not, i.e., $D_1$ is arbitrary and $C_1 = 0$ .
None of the above	The sum of the $C_1$ and $D_1$ terms is an independent solution, provided that $D_1$ satisfies Eq. (12).

As first shown by Michell in 1899,<sup>8</sup> the problem of a uniform-thickness, isotropic disk with a nonzero force resultant on a boundary requires consideration of the displacements, in order that they may be made single-valued. Similar work relating specifically to a circular annular disk was performed by Timpe in 1905.<sup>9</sup>

Timoshenko<sup>10</sup> gives Voigt credit for the first correct formulation of the equation governing plane elasticity problems in uniform-thickness anisotropic bodies. Anisotropic disks subject to body forces were treated as early as 1939 by Glushkov,<sup>11</sup> who considered a uniform-thickness centrally rotating circular disk of polar-orthotropic elastic material.

Perhaps the first analysis of the stresses in an anisotropic disk of varying thickness was made by Sen Gupta,<sup>12</sup> who in 1949 treated centrally rotating circular disks of hyperbolic and exponential profiles. So far as is known, no solutions for varying-thickness anisotropic disks subject to non-symmetric body forces have been published. A closed-form solution for the polar-orthotropic version of this problem is given in this paper.

Golecki<sup>13</sup> attributed the first analysis, which takes into account smoothly varying nonhomogeneity, to Hruban in 1944. Hruban considered the special case of an isotropic material in which only the modulus of elasticity varies with coordinate position, Poisson's ratio remaining constant. Apparently, Kovalenko<sup>14</sup> was the first to combine the effects of nonhomogeneity (varying modulus only) and varying thickness into a single quantity.

### Analysis

With the thickness assumed to be a function of the radius only and in the presence of arbitrary body forces, the polar-coordinate equilibrium equations are

$$\begin{aligned} (\partial/\partial r)(rh\sigma_r) - h\sigma_\theta + h(\partial\sigma_{r\theta}/\partial\theta) + rhF_r &= 0 \\ (1/r)(\partial/\partial r)(r^2h\sigma_{r\theta}) + h(\partial\sigma_\theta/\partial\theta) + rhF_\theta &= 0 \end{aligned} \quad (1)$$

where  $r, \theta$  are the polar coordinates,  $h$  is the thickness of the disk element,  $\sigma_r$  and  $\sigma_\theta$  are the normal stresses in the radial and tangential directions,  $\sigma_{r\theta}$  is the shear stress, and  $F_r$  and  $F_\theta$  are the radial and tangential body-force components per unit volume.

There are many possibilities for handling Eqs. (1) by an approach analogous to the stress-function method first used by Airy<sup>15</sup> for uniform-thickness plane elasticity problems without body forces. For example, for varying-thickness axisymmetric rotating disks, Föppl<sup>16</sup> used the quantities  $h\sigma_r$  and  $(h\sigma_\theta - rhF_r)$  in plane polar coordinates analogously to the way in which the rectangular-coordinate stress components were used originally in rectangular coordinates by Airy. As shown by Biot,<sup>5</sup> for example, a single body-force function is satisfactory for uniform-thickness disks subject to various kinds of body forces. However, experience has

shown that a single function is not adequate for varying-thickness disks subject to certain kinds of nonsymmetric-body forces. The concept of using two different body-force functions appears to have been originated by Kovalenko in 1955.<sup>17</sup>

The following relations among the stress components, the body-force functions, and the stress function satisfy Eqs. (1) identically:

$$h\sigma_r = \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + V_r \quad (2)$$

$$h\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} + V_\theta \quad h\sigma_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta} \left( \frac{\phi}{r} \right)$$

where the body-force functions  $V_\theta$  and  $V_r$  must be defined as follows:

$$\frac{\partial V_\theta}{\partial\theta} = rhF_\theta \quad \frac{\partial}{\partial r}(rV_r) - V_\theta = rhF_r \quad (3)$$

The condition of compatibility of strains requires that the following relation be satisfied:

$$\frac{\partial}{\partial r} \left( r \frac{\partial\epsilon_{r\theta}}{\partial\theta} - r^2 \frac{\partial\epsilon_\theta}{\partial r} \right) + r \frac{\partial\epsilon_r}{\partial r} - \frac{\partial^2\epsilon_r}{\partial\theta^2} = 0 \quad (4)$$

where  $\epsilon_r$  and  $\epsilon_\theta$  are the radial and tangential strains and  $\epsilon_{r\theta}$  is the shear strain. The following stress-strain relations for a polar-orthotropic elastic material are used:

$$\epsilon_r = \frac{e\sigma_r - \nu\sigma_\theta}{eE} \quad \epsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{eE} \quad \epsilon_{r\theta} = \frac{2c\sigma_{r\theta}}{eE} \quad (5)$$

where  $E$  is the modulus of elasticity corresponding to the radial direction,  $\nu$  the Poisson's ratio corresponding to the tangential direction,  $e$  the ratio of tangential modulus to radial modulus, and  $c$  denotes  $eE/2G$ , where  $G$  is the modulus of rigidity corresponding to  $r, \theta$ . Instead of assuming that the material is homogeneous, the modulus of elasticity is considered here to be an arbitrary, smoothly varying function of the radius only, whereas the other elastic parameters remain constant.

Following Kovalenko,<sup>14</sup> it is expedient to introduce here the quantity  $S$ , called the compliance and defined as the reciprocal of the stiffness  $Eh$ . Finally, substituting Eqs. (2) and (5) and the definition of  $S$  into Eq. (4) and simplifying gives the following result:

$$\begin{aligned} S \left[ \frac{\partial^4\phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3\phi}{\partial r^3} - \frac{e}{r^2} \frac{\partial^2\phi}{\partial r^2} + \frac{e}{r^3} \frac{\partial\phi}{\partial r} + \frac{2(c-\nu)}{r^2} \frac{\partial^4\phi}{\partial r^2\partial\theta^2} - \right. \\ \left. \frac{2(c-\nu)}{r^3} \frac{\partial^3\phi}{\partial r\partial\theta^2} + \frac{2(c-\nu+e)}{r^4} \frac{\partial^2\phi}{\partial\theta^2} + \frac{e}{r^4} \frac{\partial^4\phi}{\partial\theta^4} \right] + \\ \frac{dS}{dr} \left[ 2 \frac{\partial^3\phi}{\partial r^3} + \frac{2-\nu}{r} \frac{\partial^2\phi}{\partial r^2} - \frac{e}{r^2} \frac{\partial\phi}{\partial r} + \frac{2(c-\nu)}{r^2} \frac{\partial^3\phi}{\partial r\partial\theta^2} - \right. \\ \left. \frac{2(c-\nu)+e}{r^3} \frac{\partial^2\phi}{\partial\theta^2} \right] + \frac{d^2S}{dr^2} \left[ \frac{\partial^2\phi}{\partial r^2} - \frac{\nu}{r} \frac{\partial\phi}{\partial r} - \frac{\nu}{r^2} \frac{\partial^2\phi}{\partial\theta^2} \right] = \\ \left[ \nu \frac{\partial^2}{\partial r^2} - \frac{e-2\nu}{r} \frac{\partial}{\partial r} - \frac{e}{r^2} \frac{\partial^2}{\partial\theta^2} \right] (SV_r) - \\ \left[ \frac{\partial^2}{\partial r^2} + \frac{2+\nu}{r} \frac{\partial}{\partial r} - \frac{\nu}{r^2} \frac{\partial^2}{\partial\theta^2} \right] (SV_\theta) \quad (6) \end{aligned}$$

It is believed that Eq. (6) has not been published previously, although, for the case of isotropy, it reduces to an equation given by Kovalenko.<sup>17</sup> It is interesting to note that with only a slight modification Eq. (6) is applicable to plane-strain problems in circular members, solid or hollow, having polar-orthotropic elasticity and a modulus of elasticity which varies with the radius. For such problems, it

is necessary only to substitute the quantity  $\nu/(1 - \nu)$  for  $\nu$  everywhere it appears in Eq. (6).

Now the analysis just presented, which is applicable to circular disks with any radial distribution of stiffness, is specialized to a general power function distribution expressed by

$$1/S = 1/(S_0 r^n) \tag{7}$$

where  $S_0$  is an arbitrary constant and  $n$  is an arbitrary constant that is always positive for practical disk designs. This is a generalization to an arbitrary power of the hyperbolic thickness profile, first used by Stodola.<sup>18</sup> Substituting Eq. (7) into Eq. (6) gives the following result:

$$\begin{aligned} & \frac{\partial^4 \phi}{\partial r^4} + \frac{2(1+n)}{r} \frac{\partial^3 \phi}{\partial r^3} + \frac{n^2 - \nu n + n - e}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \\ & (e + \nu n) \frac{1-n}{r^3} \frac{\partial \phi}{\partial r} + \frac{2(c-\nu)}{r^2} \frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} - 2(c-\nu) \frac{1-n}{r^3} \times \\ & \frac{\partial^2 \phi}{\partial r \partial \theta^2} + \frac{2(c-\nu)(1-n) - e(n-2) - \nu n(n-1)}{r^4} \times \\ & \frac{\partial^2 \phi}{\partial \theta^2} + \frac{e}{r^4} \frac{\partial^4 \phi}{\partial \theta^4} = \left[ \nu \frac{\partial^2}{\partial r^2} + \frac{2\nu(n+1) - e}{r} \frac{\partial}{\partial r} - \frac{e}{r^2} \frac{\partial^2}{\partial \theta^2} + \right. \\ & \left. \frac{\nu n(n+1) - en}{r^2} \right] (V_r) - \left[ \frac{\partial^2}{\partial r^2} + \frac{2(n+1) + \nu}{r} \frac{\partial}{\partial r} - \right. \\ & \left. \frac{\nu}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{n(n+1+\nu)}{r^2} \right] (V_\theta) \tag{8} \end{aligned}$$

Rather than consider all possible complementary solutions of Eq. (8), for problems involving no dislocations or only those of the Volterra type, it is expedient to use the general form of the stress function, determined in previous work<sup>19</sup> for periodicity requirements on the stress components to be given as follows:

$$\phi(r, \theta) = C_0 \theta + C_1 r \theta \sin \theta + C_2 r \theta \cos \theta - r \theta I(r) + \psi(r, \theta) \tag{9}$$

where the  $C$ 's are constants,  $\psi$  is periodic in  $\theta$  and has continuous second-partial derivatives, and  $I(r)$  is given by

$$I(r) = (1/r) \iint r h(r) a_0(r) dr \tag{10}$$

in which  $a_0$  is the zeroth term in the Fourier-series expansion of  $F_\theta$  in terms of  $\theta$ .

Instead of discussing displacement considerations next, it is now desirable to solve Eq. (8), considering only solutions of the form of terms in Eq. (9). Considering first the term  $C_0 \theta$ , substitution shows that it satisfies the homogeneous form of Eq. (8) without restriction. Also, by inserting this term into Eq. (2), it is found that it gives rise to only a shear stress, which produces a resultant axial twisting moment.

Next the term  $C_1 r \theta \sin \theta$  is considered. Since it satisfies the following integro-differential equation,<sup>†</sup>

$$\left[ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right]_{\theta=0}^{\theta=2\pi} \Big|_{r=a} = \Sigma F_x \tag{11}$$

<sup>†</sup> Equation (11) is derived as follows. Along a concentric circular cut from the disk, the only stresses acting are  $\sigma_r$  and  $\sigma_{r\theta}$ . Thus, the resultant horizontal force is

$$\Sigma F_x = h_a \mathcal{F} [\sigma_r \cos \theta - \sigma_{r\theta} \sin \theta]_{r=a} ad \theta$$

where  $h_a$  is the thickness at radius  $a$ . Putting the expressions for  $\sigma_r$  (with  $V_r$  omitted) and  $\sigma_{r\theta}$  into this equation gives

$$\mathcal{F} \left[ \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) \cos \theta + \frac{\partial^2}{\partial r \partial \theta} \left( \frac{\phi}{r} \right) \sin \theta \right]_{r=a} d\theta = \frac{\Sigma F_x}{a}$$

Integrating by parts, evaluating certain quantities, and simplifying finally results in Eq. (11).

it produces a nonzero horizontal-force resultant  $\Sigma F_x$  on a boundary  $r = a$  of the disk. The term  $C_2 r \theta \cos \theta$  analogously gives a vertical-force resultant.

The term  $C_1 r \theta \sin \theta$  satisfies the homogeneous form of Eq. (8) only when  $n = 0$  or when  $n = 1 - (e/\nu)$ . When neither is satisfied, terms of the form  $r^{-3} \cos \theta$  remain. This suggests the possibility of combining with another term leading to a remainder of the same form in such a proportion that the homogeneous compatibility equation is satisfied by the combination. The only other term that results in periodic stresses and that has this form or remainder is  $D_1 r \ln r \cos \theta$ . However, when  $n = 0$  or when  $n = (1 + e + 2c - 2\nu)/(1 - \nu)$ , the remainder vanishes. Thus, under these latter conditions  $D_1 r \ln r \cos \theta$  is an independent complementary solution of Eq. (8). These cases, namely,  $n = 0$ ,  $n = 1 - (e/\nu)$ , and  $n = (1 + e + 2c - 2\nu)/(1 - \nu)$ , are the exceptional cases; when none of these are met, the sum of the  $C_1$  and  $D_1$  terms is a complementary solution, provided that

$$D_1 = \frac{-2(e - \nu + \nu n)}{1 + e + 2(c - \nu) - (1 - \nu)n} C_1 \tag{12}$$

It is important to mention that the term  $r \ln r \cos \theta$ , like the term  $r \theta \sin \theta$ , gives a nonzero horizontal-force resultant on a boundary, since, when it is substituted into the left-hand side of Eq. (11), a nonzero constant ( $2\pi$ ) remains. Thus, if either of these terms appears in  $\phi$ , there is a nonzero horizontal-force resultant on a boundary. A nonzero vertical-force resultant is handled analogously by the terms  $r \ln r \sin \theta$  and  $r \theta \cos \theta$ .

Recapitulating, the  $C_1$  and  $D_1$  terms are complementary solutions of Eq. (8) for the cases listed, which include all possible values of  $n$ , as shown in Table 1.

Substitution of periodic terms denoted by  $\psi$  in the series form

$$\psi(r, \theta) = \sum_{p=0}^{\infty} R_p(r) \cos p \theta \tag{13}$$

leads to the same result as that obtained by using the analogous sine series. Thus, for brevity, only the cosine series is written.

Provided that there are no multiple roots, i.e., repeated roots, the general expression for  $R_p$  is as follows:

$$R_p = A_{p_1} r^{\lambda_{p_1}} + A_{p_2} r^{\lambda_{p_2}} + A_{p_3} r^{\lambda_{p_3}} + A_{p_4} r^{\lambda_{p_4}} \tag{14}$$

where the  $A_p$ 's are arbitrary constants that must be determined from the boundary conditions of the particular problem, and the  $\lambda_p$ 's are given by

$$\lambda_{p_{1,3}} = -\frac{n-2}{2} \pm \beta_{p_1} \quad \lambda_{p_{2,4}} = -\frac{n-2}{2} \pm \beta_{p_2} \tag{15}$$

and

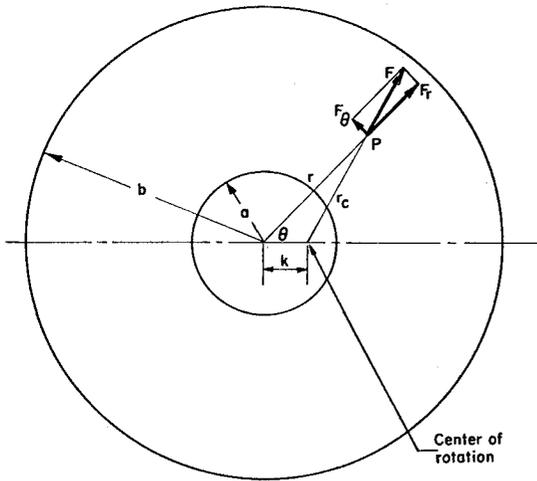
$$\beta_{p_1}^2 + \beta_{p_2}^2 = \frac{1}{2}(n-2)^2 + (1+\nu)n + e - 1 + 2(c-\nu)p^2 \tag{16}$$

$$\begin{aligned} (\beta_{p_1}^2 - \beta_{p_2}^2)^2 &= [(1+\nu)n + e - 1]^2 - \\ &4np^2[\nu n + e - \nu + (1-\nu)(c-\nu)] + \\ &4p^2[(e+1)(c-\nu) + (c-\nu)^2 p^2 + 2e - ep^2] \tag{17} \end{aligned}$$

In the case of multiple roots, assuming that  $\lambda_{p_1} = \lambda_{p_2}$  and that  $\lambda_{p_3} = \lambda_{p_4}$ , the expression for  $R_p$  becomes

$$R_p = A_{p_1} r^{\lambda_{p_1}} + A_{p_2} \lambda_{p_1} \ln r + A_{p_3} r^{\lambda_{p_3}} + A_{p_4} r^{\lambda_{p_3}} \ln r \tag{18}$$

The term  $-r \theta I(r)$  involves  $a_0$  (the first term in the Fourier-series expansion of  $F_\theta$ ), which is directly related to  $V_\theta$ , which in turn appears only in the nonhomogeneous form of Eq. (8). Thus, this term is in fact a particular solution rather than a complementary one. A class of problem in which this term would not vanish is in a problem involving a disk under-



**Fig. 1 Geometrical relationships and directions of forces for a typical point P in an eccentrically rotating disk**

going rotational acceleration about its normal axis. Then this term would be equilibrated by the  $C_0\theta$  term.

To obtain a general expression for the particular solution of Eq. (8), it is convenient to use a Fourier power series to approximate the body-force functions:

$$V_r = \sum_i \sum_j C_{ij} r^i \cos j\theta \quad V_\theta = \sum_s \sum_t C_{st} r^s \cos t\theta \quad (19)$$

plus similar sine terms. Then, the particular solution takes the form

$$\phi_p = \sum_i \sum_j B_{ij} C_{ij} r^{2+i} \cos j\theta + \sum_s \sum_t B_{st} C_{st} r^{2+s} \cos t\theta \quad (20)$$

where the constants  $B_{ij}$  and  $B_{st}$  are found by substitution to be given by

$$B_{ij} K_{ij} = \nu(n^2 + n + 2ni + i + i^2) - e(n + i - j^2) \quad (21)$$

and

$$B_{st} K_{st} = -n^2 - (1 + \nu + 2s)n + s^2 - (1 + \nu)s - \nu t^2 \quad (22)$$

where

$$K_{ij} = (2 + i)(1 + i)[n^2 + (1 - \nu + 2i)n + i^2 + i - e] + (e + \nu n)(1 - n)(2 + i) - 2(c - \nu)[(n + i)(2 + i) + 1 - n]j^2 + [e(n - 2) + \nu n(n - 1) + ej^2]j^2 \quad (23)$$

and  $K_{st}$  is given by the same expression with  $s, t$  replacing  $i, j$ . The particular solution breaks down under certain conditions, for example, when  $K_{ij} = 0$  or  $K_{st} = 0$ . Two cases of importance in which this happens are 1) when a body-force function ( $V_r$  or  $V_\theta$ ) is constant ( $i = j = 0$  or  $s = t = 0$ ) and either  $n = 0$  or  $n = (e - 1)/(1 - \nu)$ ; and 2) when the disk is uniform thickness ( $n = 0$ ) and isotropic ( $e = c - \nu = 1$ ), and a body-force function is harmonic ( $i = j$  or  $s = t$ ).

For all cases involving a constant body-force function, the corresponding particular solution is zero, so that  $\phi_p$  given by Eq. (20) is not valid for any case involving constant body-force functions, regardless of the thickness-variation factor  $n$ . As for case 2, it was noted by Biot<sup>5</sup> that the particular solution for the uniform-thickness, isotropic case vanishes provided that the body forces are derivable from a harmonic function  $V$ . Thus, for case 2, the particular solution is zero.

**Application to a Disk Rotating about an Eccentric Normal Axis**

The problem is to determine the stresses in a circular disk mounted concentrically on a circular shaft rotating at constant speed about an axis parallel with, but eccentric to, the axis of the shaft and disk. This problem is encountered when

clearances in shaft-support bearings are excessive or when the shaft is sufficiently flexible to permit significant deflection.<sup>§</sup> This problem has not been treated previously even for the uniform-thickness, isotropic case, since published solutions<sup>6,20-22</sup> for eccentrically rotating disks treat only solid disks with a centripetal force concentrated at the rotational center.

Figure 1 shows an axial view of the disk and the geometrical parameters involved. Also shown in Fig. 1 is the direction of the centrifugal body-force resultant, which has a magnitude per unit volume of

$$F = \gamma\Omega^2 r_c/g \quad (24)$$

where  $\gamma$  is the specific weight,  $g$  the gravitational acceleration,  $\Omega$  the rotational velocity, and  $r_c$  the distance from the center. The respective radial and tangential components of  $F$  are found to be given by

$$F_r = (\gamma\Omega^2/g)(r - k \cos\theta) \quad (25)$$

$$F_\theta = (\gamma\Omega^2/g)(k \sin\theta) \quad (26)$$

where  $k$  is the eccentricity. For a concentrically rotating disk,  $k = 0$ ,  $F_r$  is independent of  $\theta$ , and  $F_\theta = 0$ .

Since, for this problem, there is no traction on the outer boundary, the term  $C_0\theta$  is eliminated. Furthermore, since the vertical-force resultant is zero, terms of the forms  $r \ln r \sin\theta$  and  $r\theta \cos\theta$  are omitted. Symmetry about the horizontal axis requires that all sinusoidal terms in  $\theta$  or its multiples be excluded. As can be seen in Eq. (26), the zeroth term is not present in the Fourier-series expansion of  $F_\theta$ ; thus  $a_0 = 0$  and  $I(r) = 0$  from Eq. (10). Thus, it is seen that the only terms of the complementary solution which need be considered are of the form  $C_1 r\theta \sin\theta$ ,  $D_1 r \ln r \cos\theta$ ,  $R_0(r)$ , and  $R_1(r)$ . The specific values of  $C_1$  and  $D_1$ , as well as the specific forms of  $R_0$  and  $R_1$ , depend upon the nature of the elastic stiffness variation.

Using Eqs. (25) and (26), Eqs. (3) are integrated to give

$$V_\theta = -\frac{\gamma\Omega^2 h_0}{g} k r^{1-n} \cos\theta + X_r(r) \quad (27)$$

$$V_r = \frac{\gamma\Omega^2 h_0}{g} \left( \frac{r^{2-n}}{3-n} - \frac{2k}{2-n} r^{1-n} \cos\theta \right) + \frac{1}{r} \int X_r(r) dr + \frac{1}{r} X_\theta(\theta) \quad (28)$$

where  $X_\theta$  and  $X_r$  are functions of integration. Particular solutions resulting from  $V_r$  and  $V_\theta$  depend upon the details of the disk, as illustrated in the following examples.

**Example 1: Isotropic, Uniform-Thickness Disk**

In the uniform-thickness case, it is always possible to make the two body-force functions identical by proper selection of  $X_\theta$  and  $X_r$ . To do so in the present case, it is found that  $X_\theta = 0$  and  $X_r = \gamma\Omega^2 h_0 r^2/2g$ . Then

$$V_\theta = V_r = (\gamma\Omega^2 h_0/g) (\frac{1}{2} r^2 - kr \cos\theta)$$

and, using Eqs. (20-23) and the discussion following them for case 2, the particular solution is found to be given by

$$\phi_p = [(2 - \nu)/16](\gamma\Omega^2 h_0/g) r^4$$

After applying Eqs. (15-17) and noting the double roots, the complementary solution can be written as

$$\phi_c/Ch_0 = A_{01} + A_{02} \ln r + A_{03} r^2 + A_{04} r^2 \ln r + (A_{11} r^{-1} + A_{12} r + A_{13} r^3 + A_{14} r \ln r) \cos\theta + C_1 r\theta \sin\theta \quad (29)$$

<sup>§</sup> If the disk overhangs the bearings or is nonsymmetrically located with respect to them, it also undergoes a rotation and thus is subject to bending stresses that are not considered here.

where  $C$  is an arbitrary constant. Consideration of the stresses corresponding to  $\phi_2$  in Eq. (29) shows that  $A_{01}$  and  $A_{12}$  do not affect the stresses and thus may be omitted. A study of the displacements shows that, in order for them to be single-valued,  $A_{04} = 0$  and  $A_{14} = -(1 - \nu)C_1/2$ . Thus, the complete stress function now can be written as

$$\frac{\phi}{Ch_0} = A_{02} \ln r + A_{03}r^2 - \frac{2 - \nu}{16} r^4 + (A_{11}r^{-1} + A_{13}r^3) \cos \theta + C_1 r (\theta \sin \theta - \frac{1 - \nu}{2} \ln r \cos \theta) \quad (30)$$

where  $C = \gamma\Omega^2/g$  for convenience.

To evaluate the constants of integration in Eq. (30), the physical boundary conditions must be specified. Assuming the outer periphery ( $r = b$ ) to be a free edge, both the radial and the shear stress along it are zero. As is often the practice in manufacture, the disk is assumed to be fitted to the shaft by means of a press or shrink fit. Thus, under static conditions (i.e., no rotational velocity), the radial stress at the inner boundary is equal to the negative of the contact pressure, assumed to be uniform around the circumference. Since the problem of the contact stresses in a thin disk and a long elastic shaft is a three-dimensional one, any assumptions made as to deformations in two directions only are understood to be approximate. Therefore, it is assumed here that the radial-pressure distribution consists of a uniform pressure  $p$ , conservatively assumed to be equal to the static contact pressure<sup>11</sup> with a superimposed pressure increment, due to centripetal action, proportional to the cosine of the angle  $\theta$  in Fig. 1. Then the radial-stress distribution at the inner boundary is given by

$$(\sigma_r)_{r=a} = -p(1 + C_p \cos \theta) \quad (31)$$

where  $C_p$  is a dimensionless constant always less than unity. As a second boundary condition at the inner radius, it is assumed that  $\sigma_{r\theta}$  there is zero, the friction along this edge being taken as negligible. These last two conditions were used by Bickley<sup>23</sup> in his analysis of the related stress problem for a plate with a hole containing an oversized rivet or pin pulled in the plane of the plate.

The pressure coefficient  $C_p$  is evaluated by equating the resultant centrifugal force to the resultant force at the inner boundary to give

$$C_p = \gamma\Omega^2 k V_0 / \pi g p a h_a \quad (32)$$

where  $V_0$  is the disk volume and  $h_a$  is the disk thickness at the inner radius. For the uniform-thickness disk,

$$C_p = (Ckb/p) [(1 - \alpha^2)/\alpha] \quad (33)$$

where  $\alpha$  denotes the ratio  $a/b$ .

Substitution of the unevaluated terms of  $\phi$  in Eq. (30) into the four boundary conditions results in two equations in the constants  $A_{02}$  and  $A_{03}$  and four equations in the constants  $A_{11}$  and  $A_{13}$ . However, the latter four equations have two different pairs of identical left-hand sides. Now the four equations can be compatible only if the terms appearing in their right-hand sides are related in such a way that the appropriate right-hand sides are identical. This results in a relationship that is a consequence of static equilibrium<sup>9</sup> and that can be used to determine the constant  $C_1$ . The resulting expressions for the constants are as follows:

$$A_{02} = \left( \frac{\nu}{4} - \frac{1}{1 - \alpha^2} \frac{p}{Cb^2} \right) \alpha^2 b^4$$

$$A_{11} = - \frac{1 - \nu}{8} \frac{kb^4}{1 + 1/\alpha^2}$$

|| Actually, the contact pressure changes during operation because of the differences between the centrifugal expansions of the disk and of the shaft.

$$A_{03} = \left[ \nu(1 + \alpha^2) + \frac{4\alpha^2}{1 - \alpha^2} \frac{p}{Cb^2} \right] \frac{b^2}{8}$$

$$A_{13} = - \frac{1 - \nu}{8} \frac{k}{1 - \alpha^2} \quad C_1 = \frac{1}{2} kb^2$$

Insertion of these values into  $\phi$  and thence into Eq. (2) gives the stress distributions for various values of the quantity  $Cb^2$ , eccentricity  $k$ , and contact pressure  $p$ . For example, assuming that  $p$  is the minimum required to prevent the disk from loosening on the shaft under operation with an eccentricity,  $C_p = 1$ , and thus  $p/Cb^2 = k(1 - \alpha^2)/b\alpha$ . Then the tangential-stress distribution is given by the following expression:

$$\frac{\sigma_\theta}{Cb^2} = - \left( \frac{\nu}{4} \alpha^2 - \frac{k}{b} \alpha \right) \left( \frac{b}{r} \right)^2 + \frac{\nu}{4} (1 + \alpha^2) + \frac{k}{b} \alpha - \left( \frac{4 - 3\nu}{4} \right) \left( \frac{r}{b} \right)^2 - \left[ \frac{(b/r)^3}{1 + 1/\alpha^2} + \frac{b}{r} + \frac{2 + \alpha^2}{1 - \alpha^2} \left( \frac{r}{b} \right) \right] \frac{1 - \nu}{4} \frac{k}{b} \cos \theta$$

In particular, for  $\nu = \frac{1}{3}$ ,  $\alpha = 0.1$ , and  $k/b = 0.001$ , the maximum tangential stress at the inner radius is  $0.0688 Cb^2$ . For a steel disk ( $\gamma = 0.283$  lb/in.<sup>3</sup>), 30 in. in diameter ( $b = 15$  in.) and running at 6000 rpm ( $\Omega = 628$  rad/sec), this amounts to a stress of only 1790 psi.

It is noted that, when boundary values vary with  $\cos \theta$  (i.e., for  $p = 1$ ) in uniform-thickness disks with arbitrary values of the elastic coefficients  $e, c$ , and  $\nu$ , the complementary solution  $\phi_c$  always contains terms of the form  $r \cos \theta$  and  $r \ln r \cos \theta$ , since then unity is a double root of the characteristic equation.

**Example 2: Polar-Orthotropic, Varying-Thickness Disk**

As an example, an homogeneous disk with thickness varying inversely with the radius (i.e.,  $n = 1$ ), an orthotropic ratio  $e$  of  $\frac{1}{4}$ , and a Poisson's ratio  $\nu$  of  $\frac{1}{2}$  is treated. Although a value of  $\frac{1}{2}$  is the upper bound for  $\nu$  of an isotropic material, for a polar-orthotropic material with  $e < 1$ , this value is reasonable. Using an equation proposed by Lang<sup>24</sup> to estimate the shear modulus, coefficient  $c$  is equal to  $\frac{5}{8}$ .

Taking the functions  $X_\theta$  and  $X_r$ , each to be zero for simplicity, the body-force functions are found to be, by Eqs. (27) and (28),

$$V_\theta = -Ch_0k \cos \theta \quad V_r = Ch_0(\frac{1}{2}r - 2k \cos \theta)$$

where  $C$  is  $\gamma\Omega^2/g$ . In a manner identical to that used for the uniform-thickness, isotropic case, the complete solution is found to be given by

$$\phi/C = A_{01}r^{-1/2} + A_{02} + A_{03}r^{3/2} + A_{04}r + (A_{11}r^{-0.618} + A_{12} + A_{13}r^{1.618} + A_{14}r) \cos \theta + C_1 r (\theta \sin \theta - \frac{1}{2} \ln r \cos \theta) + h_0 \left( \frac{5}{12} \frac{r^3}{b} + \frac{1}{2} k r^2 \cos \theta \right)$$

As in the case of uniform-thickness, isotropic disks, two of the coefficients, here  $A_{02}$  and  $A_{14}$ , do not affect the stresses and thus may be set equal to zero. Also, an investigation of the displacements shows that, in order for them to be single-valued, both  $A_{04}$  and  $A_{12}$  must be equal to zero.

Using the same boundary conditions as before, the constants of integration are evaluated in the same manner, with the following results:

$$A_{01} = \left[ \frac{p}{Cb^2} \alpha^{3/2} - \frac{13}{21} \alpha^2 (1 - \alpha^{3/2}) \right] \frac{2b^{7/2}}{1 - \alpha^2}$$

$$A_{03} = - \left[ \frac{p}{Cb^2} \alpha^{3/2} - \frac{13}{21} (1 - \alpha^{7/2}) \right] \frac{2b^{3/2}}{3(1 - \alpha^2)}$$

$$A_{11} = \frac{kb^{2.618}}{1.618\alpha} \frac{1 - \frac{1}{2}\alpha - (\frac{3}{2} - \alpha)\alpha^{0.618}}{1 - \alpha^{-3}}$$

$$A_{13} = \frac{kb^{0.382}}{0.618\alpha^{0.618}} \frac{1 - \frac{1}{2}\alpha - (\frac{3}{2} - \alpha)\alpha^{-1.618}}{1 - \alpha^{-2.236}}$$

$$c_1 = kb$$

Using the same assumption as before for the magnitude of the shrink-fit pressure, the tangential-stress distribution can be expressed as follows:

$$\frac{\sigma_\theta}{Cb^2} = \left[ 2 \frac{k}{b} (1 - \alpha)\alpha^{3/2} - \frac{13}{21} \alpha^2 (1 - \alpha^{3/2}) \right] \frac{3}{2} \frac{(b/r)^{3/2}}{1 - \alpha^2} - \left[ 2 \frac{k}{b} (1 - \alpha)\alpha^{3/2} - \frac{13}{21} (1 - \alpha^{7/2}) \right] \frac{1}{2} \frac{(r/b)^{1/2}}{1 - \alpha^2} + \frac{5}{21} \left( \frac{r}{b} \right)^2 + \left[ \frac{0.990}{b^2} (A_{11}r^{-1.618} + A_{13}r^{0.618}) - \frac{1}{2} \frac{k}{b} \right] \cos\theta$$

In particular, for a disk having the same  $\alpha$  and  $k/b$  values as before, the highest tangential stress at the inner radius is equal to 0.01414  $Cb^2$ , or 3680 psi for a disk of the same density and outside diameter and running at the same speed. This is more than twice the previous value, so that it is seen that increasing the relative radial stiffness is not beneficial under the loading conditions assumed here. It also is interesting to note that in both cases the disk basic thickness  $h_0$  does not affect the stress value.

### Discussion

The similarity between the varying-thickness, polar-orthotropic case just considered and the uniform-thickness, isotropic case is remarkable. However, it should be pointed out that, for many combinations of thickness variation and polar orthotropy, only two terms, instead of four, disappear from the original complementary solution because of considerations of null effectiveness of stresses and/or uniqueness of displacements. This type of problem can be illustrated easily by the case of a disk mounted on a perfectly rigid shaft. However, the displacement of the disk-shaft boundary must not be taken to be zero, since under loading the disk centerline undergoes a rigid-body motion consisting of a translation, denoted by  $\Delta$ , in the direction of the resultant centrifugal force but no rotation. Then the radial, tangential, and rotational displacements of all points on the disk-shaft intersection are as follows:

$$u = -\Delta \cos\theta \quad v = \Delta \sin\theta \quad \omega = 0 \quad (34)$$

From these and the general expressions for the displacements due to the various types of stress functions,<sup>25</sup> it can be shown that three relations remain. These relations provide a means for determining an unknown displacement constant of integration as well as the two unknown stress-function coefficients.

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